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# Clifford lattices and a conformal generalization of Desargues' theorem

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## Abstract

Lattices composed of Clifford point-circle configurations provide a geometric representation of the discrete Schwarzian KP (dSKP) equation. Based on an  $A_n$  perspective on such lattices, it is shown that their integrability, and hence that of the dSKP equation, is a consequence of a conformal generalization of the classical Desargues Theorem of projective geometry.

*Keywords:* conformal geometry, integrable systems, Desargues Theorem, Clifford configuration

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## 1. Introduction

The celebrated Hirota-Miwa equation [8] constitutes a discrete ‘master equation’ in soliton theory [1] and is known to come in various guises. The integrability of any of these avatars, regarded as a 6-point equation defined on a three-dimensional lattice of  $\mathbb{Z}^3$  combinatorics, is reflected by the fact that its domain may be extended to a  $\mathbb{Z}^4$  lattice in such a way that the equation holds on any of the four systems of parallel coordinate hyperplanes.

The central perspective of this paper is that the domain of the equation may be more symmetrically considered as an  $A_3$  (root) lattice

$$\{(m_0, m_1, m_2, m_3) \in \mathbb{Z}^4 : \sum_i m_i = \text{const}\}, \quad (1.1)$$

that is, a three-dimensional lattice with four natural (linearly related) coordinates. The integrability may then be encapsulated by a cellular consistency condition for extending the domain to an  $A_4$  lattice in such a way that 6-point equations hold on the *five* natural systems of parallel  $A_3$ -type coordinate hyperplanes. This  $A_n$  perspective has recently been embraced in the classification of integrable 6-point equations of Hirota-Miwa type by Adler, Bobenko & Suris [2]. As noted by Doliwa [6], the  $A_n$  lattice also emerges as a natural domain when relating quadrilateral (or Darboux) lattices to ‘Desargues lattices’ by means of discrete Laplace transformations.

The ‘Schwarzian’ avatar of the Hirota-Miwa equation is the (scalar) discrete Schwarzian Kadomtsev-Petviashvili (dSKP) equation [7, Eq 30]

$$\frac{(q_1 - q_{12})(q_2 - q_{23})(q_3 - q_{13})}{(q_{12} - q_2)(q_{23} - q_3)(q_{13} - q_1)} = -1. \quad (1.2)$$

This equation’s natural algebraic interpretation relates 6 points in the projective line  $P^1$  in such a way that any five points uniquely determine the sixth. By interpreting the indices as coordinate shifts in the  $\mathbb{Z}^3$  lattice, e.g.

$$q_{13}(m_1, m_2, m_3) = q(m_1 + 1, m_2, m_3 + 1),$$

the equation (1.2), applied to a lattice function  $q: \mathbb{Z}^3 \rightarrow P^1$ , imposes the condition on 6 of the 8 points of each unit cube.

In the complex case, we may identify  $\mathbb{CP}^1$  with the conformal 2-sphere  $\mathbb{R}Q^2$  and it has been shown [10, 9] that then the equation is also related to Clifford’s first point-circle theorem [3] in conformal 2-space  $Q^2$  (see Section 2 for details). Thus the dSKP equation has a geometric interpretation in terms of ‘Clifford lattices’ in  $Q^2$ . A similar connection may be found in the quaternionic case, leading to generalized Clifford configurations in conformal 4-space, as discussed in [12].

In this paper, we present a novel conformal generalization of the classical Desargues Theorem and demonstrate that it may be regarded as the geometric source of the integrability of Clifford lattices, and thus as a geometric reflection of the integrability of the dSKP equation. In the special case that a Clifford lattice is a Menelaus lattice [10], it is the classical Desargues Theorem that is the geometric source of the integrability (see Remark 6.3). Indeed, it is this that leads Doliwa [6] to use the term ‘Desargues lattices’ for the natural projective version of Menelaus lattices.

## 2. Clifford and Miquel configurations

In this paper, for purposes of familiarity and simplicity, conformal 2-space  $Q^2$  will be considered just over the field  $\mathbb{R}$  and so will be the conformal 2-sphere, which may be projected stereographically onto the Euclidean plane. In this case, there is a unique circle  $C(x, y, z)$  through any three distinct points  $x, y, z \in Q^2$ . To be more general,  $C(x, y, z)$  would have to be a hyperplane section of a smooth quadric  $Q^2 \subset P^3$  through three sufficiently general points. In any case, we will not explicitly consider degenerate cases of results that follow (e.g. Clifford's Theorem) and thus will implicitly assume that all sets of points are sufficiently general.

A *Clifford lattice* is a configuration in  $Q^2$  given by a map  $q: \mathbb{Z}^3 \rightarrow Q^2$  with the property that the six shifts  $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$  satisfy the two equivalent conditions of Clifford's Theorem:

**Theorem 2.1** (Clifford). *Let  $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$  be six points in  $Q^2$ . Then the four circles  $C(q_1, q_2, q_3)$ ,  $C(q_1, q_{12}, q_{13})$ ,  $C(q_2, q_{12}, q_{23})$ ,  $C(q_3, q_{13}, q_{23})$  pass through a common point  $c^{**}$  if and only if the four circles  $C(q_1, q_2, q_{12})$ ,  $C(q_1, q_3, q_{13})$ ,  $C(q_2, q_3, q_{23})$ ,  $C(q_{12}, q_{13}, q_{23})$  pass through a common point  $c^*$ .*

When these two equivalent conditions are satisfied, we will say that the six points 'satisfy Clifford's condition' or 'form a Clifford configuration' (Figure 1).

**Remark 2.2.** One important consequence of Clifford's Theorem is that a Clifford configuration has 'antipodal symmetry', that is, if the points  $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$  are relabelled  $q_{23}, q_{13}, q_{12}, q_3, q_2, q_1$ , by taking complementary indices, then they still satisfy Clifford's condition. Indeed, the first and second conditions in Clifford's Theorem are exactly interchanged by this change of labelling. Note that this antipodal symmetry is also transparent in (1.2), because the symmetry interchanges the numerator and denominator of the left-hand-side, but  $1/(-1) = -1$ .

Clifford's Theorem is an immediate corollary of the following formulation of Miquel's Theorem (see Figure 2):

**Theorem 2.3** (Miquel). *Let  $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$  be six points in  $Q^2$ . Then the three circles  $C(q_1, q_{12}, q_{13})$ ,  $C(q_2, q_{12}, q_{23})$ ,  $C(q_3, q_{13}, q_{23})$  pass through a common point if and only if the three circles  $C(q_1, q_2, q_{12})$ ,  $C(q_1, q_3, q_{13})$ ,  $C(q_2, q_3, q_{23})$  pass through a common point.*

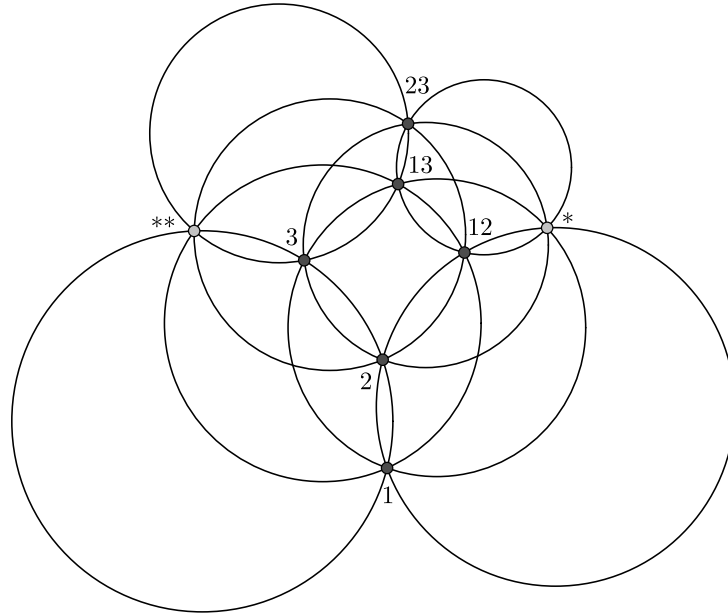


Figure 1: A Clifford configuration

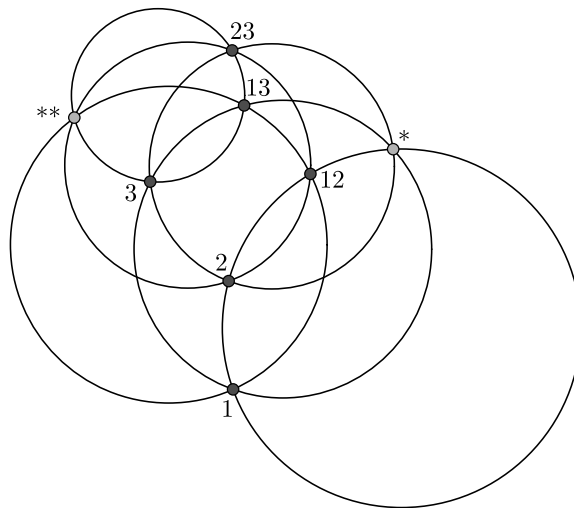


Figure 2: A Miquel configuration

To see how Miquel's Theorem implies Clifford's, note that, while the natural symmetry group of Miquel's Theorem is  $S_3$ , permuting the three symbols 1,2,3, the natural symmetry group of Clifford's Theorem is  $S_4$ , with the six points more symmetrically labelled  $q_{01}, q_{02}, q_{03}, q_{12}, q_{13}, q_{23}$ . Dropping any one of the symbols 0,1,2,3 yields an instance of Miquel's Theorem using the remaining three symbols. On the other hand, the common point of any three of the four circles in Clifford's Theorem is the second intersection point of any two of them and hence all four such common points are seen to coincide as soon as they are proved to exist.

### 3. The $A_n$ perspective

Because the symmetry group of a Clifford configuration is  $S_4$ , it is natural to consider the domain of a Clifford lattice to be an  $A_3$  (root) lattice. More generally, the  $A_n$  lattice is

$$\{(m_0, \dots, m_n) \in \mathbb{Z}^{n+1} : \sum_i m_i = c\}$$

for some constant  $c$ , whose value is essentially irrelevant. The key feature is that an  $A_n$  lattice has  $n+1$  natural coordinates, any one of which can be dropped to give a (non-canonical) isomorphism with  $\mathbb{Z}^n$ . The coordinates are naturally permuted by  $S_{n+1}$  and thus the full symmetry group of the lattice, including coordinate shifts, is  $\mathbb{Z}^n \rtimes S_{n+1}$ , which is an affine Coxeter group of type  $\tilde{A}_n$ . Indeed, the  $A_n$  lattice may also be considered as a 'honeycomb', in the sense of Coxeter [4], that is, an affine (as opposed to spherical) polytope associated with this Coxeter group.

Just as  $\mathbb{Z}^n$  can be divided into  $n$ -cube cells by restricting each of the  $n$  coordinates to two successive integer values, so an  $A_n$  lattice has a cell decomposition by making the same restriction on each of the  $n+1$  coordinates. However, now the cells are of several different types, which we denote by  $g_k^{n+1}$  for  $k = 1, \dots, n$ . These are obtained by slicing an  $n$ -cube cell given by  $n$  of the coordinates by the integral hyperplanes of the remaining coordinate.

For example, a 3-cube is sliced into two tetrahedra  $g_1^4$  and  $g_3^4$ , and one octahedron  $g_2^4$  (see Figure 3). The vertices of such an octahedral  $g_2^4$  cell are the natural labels of a Clifford configuration, namely 1, 2, 3, 12, 13, 23 in  $\mathbb{Z}^3$  notation, or 01, 02, 03, 12, 13, 23 in  $A_3$  notation.

In general, the vertices of the typical  $g_k^{n+1}$  cell can be labelled by all choices of  $k$  symbols from  $0, \dots, n$ , indicating that they may be obtained

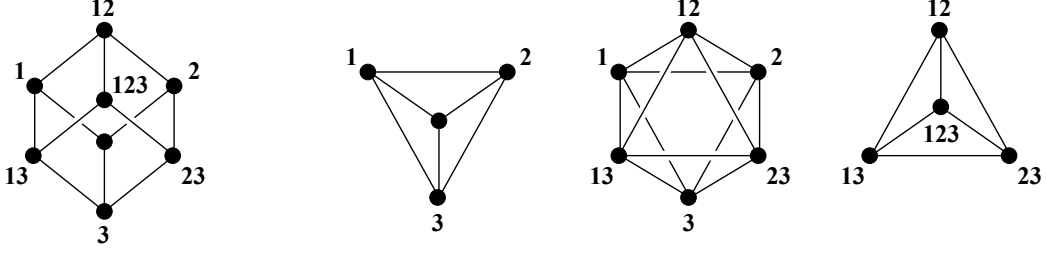


Figure 3: The cells of  $\mathbb{Z}^3$  and  $A_3$  lattices

from a single point  $p \in \mathbb{Z}^{n+1}$  by applying all possible sets of  $k$  distinct coordinate shifts. Note that all such shifts will lie in the same  $A_n$  lattice, although the original point  $p$  will not. We call this the  $A_n$  notation for these points. Dropping any one of the symbols  $0, \dots, n$  (usually 0) we get a  $\mathbb{Z}^n$  notation for the same points.

Of special interest below are the two four-dimensional cells  $g_2^5$  and  $g_3^5$ . These each have two sorts of faces: the first having five tetrahedral  $g_1^4$ s and five octahedral  $g_2^4$ s, the second having five octahedral  $g_2^4$ s and five tetrahedral  $g_3^4$ s. Figure 4 shows the octahedral faces of a pair of adjacent  $g_2^5$  and  $g_3^5$  cells, in  $\mathbb{Z}^4$  notation; notice how they meet along one common octahedral face.

#### 4. A Conformal Desargues Theorem

We now present our main theorem that we will show, in the next section, to be the geometric key to the integrability of Clifford lattices.

**Theorem 4.1.** *Let  $q_{01}, \dots, q_{34}$  be a configuration of ten points in  $Q^2$  labelled by the vertices of a  $g_2^5$  cell (in  $A_4$  notation). If three of the octahedral faces, e.g. those in the first row of Figure 4, form Clifford configurations, then so do the other two, e.g. those in the second row of Figure 4.*

*Proof.* The five octahedra under consideration are those with vertices labelled by the six pairs formed from four of the labels  $0, 1, 2, 3, 4$ , i.e. by omitting each of the labels in turn. Thus we may suppose that we have Clifford points  $c_4^*, c_3^*, c_2^*$  for the three octahedra that omit the labels  $4, 3, 2$  and we must prove the existence of Clifford points  $c_1^*, c_0^*$  for the other two octahedra.

The final configuration (see Figure 5) should consist of the ten circles  $C(q_{ij}, q_{jk}, q_{ik})$  each also passing through  $c_l^*$  and  $c_m^*$ , where  $i, j, k, l, m$  are the labels  $0, 1, 2, 3, 4$  in some order. So we may begin by assuming this holds whenever  $l, m$  are two of  $2, 3, 4$ .

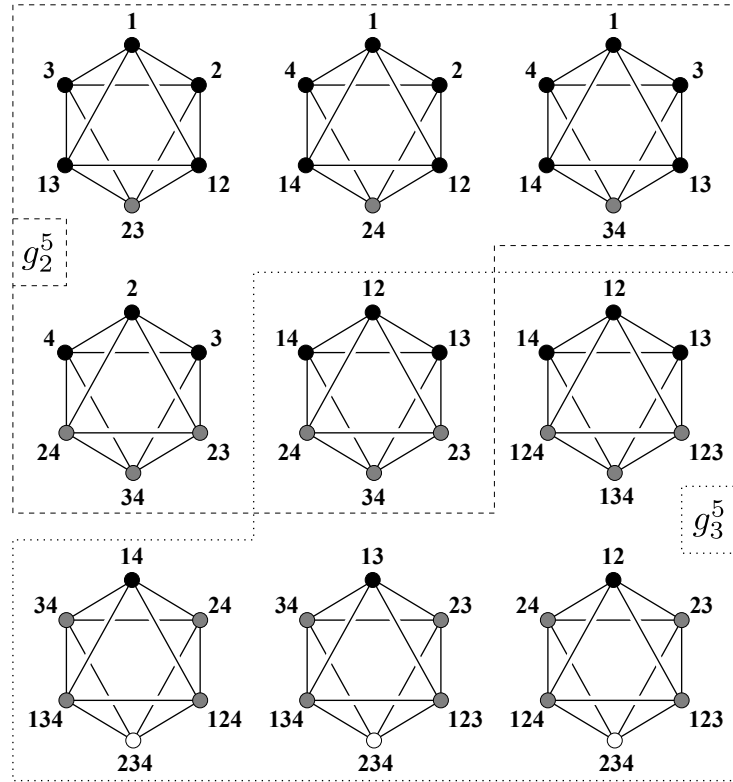


Figure 4: The nine octahedral faces of the  $g_2^5$  and  $g_3^5$  cells (in  $\mathbb{Z}^4$  notation)



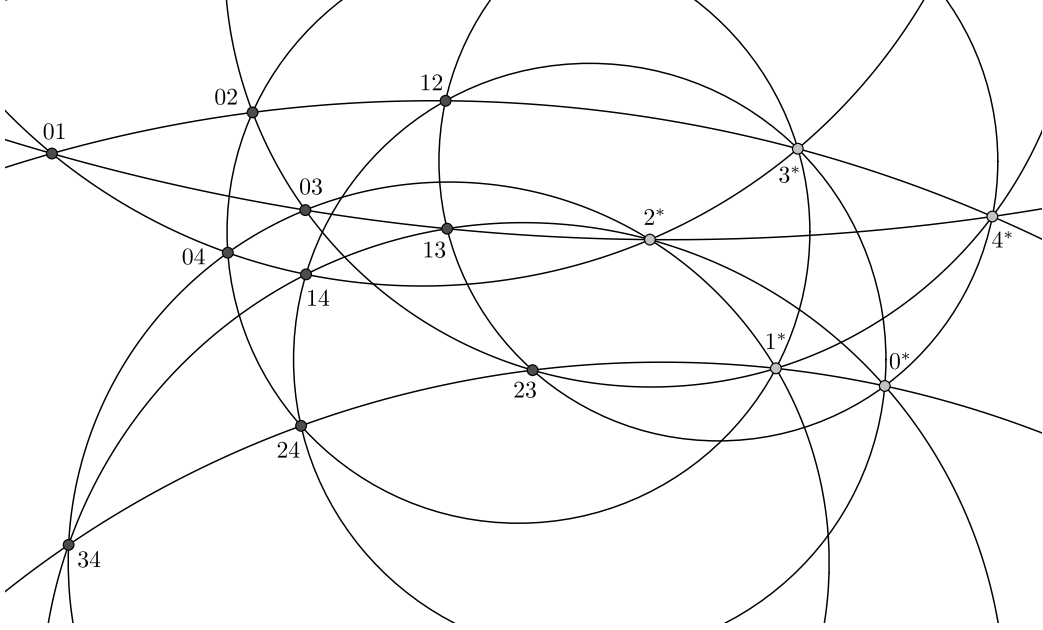


Figure 5: A conformal Desargues configuration

From the three assumed Clifford configurations, we know that

$$\begin{aligned} C(q_{12}, q_{13}, q_{23}) &= C(q_{12}, q_{13}, c_4^*) \\ C(q_{12}, q_{14}, q_{24}) &= C(q_{12}, q_{14}, c_3^*) \\ C(q_{13}, q_{14}, q_{34}) &= C(q_{13}, q_{14}, c_2^*) \end{aligned}$$

and we can then use Miquel's Theorem to deduce that these three circles have a common point, which we call  $c_0^*$ , because the three circles  $C(q_{12}, c_3^*, c_4^*)$ ,  $C(q_{13}, c_2^*, c_4^*)$ ,  $C(q_{14}, c_2^*, c_3^*)$  have a common point, namely  $q_{01}$ .

Reversing the role of 0 and 1, we obtain a point  $c_1^*$ , through which the three circles  $C(q_{02}, q_{03}, q_{23})$ ,  $C(q_{02}, q_{04}, q_{24})$ ,  $C(q_{03}, q_{04}, q_{34})$  pass. It remains to show that  $c_0^*$  and  $c_1^*$  both lie on the circle  $C(q_{24}, q_{34}, q_{23})$ .

Using Miquel's Theorem again, observe that the circles  $C(q_{14}, q_{24}, c_0^*)$ ,  $C(q_{04}, q_{24}, c_1^*)$ ,  $C(q_{04}, q_{14}, c_2^*)$  have a common point, namely  $c_3^*$ , and hence the circles  $C(q_{04}, c_1^*, c_2^*)$ ,  $C(q_{14}, c_0^*, c_2^*)$ ,  $C(q_{24}, c_0^*, c_1^*)$  also have a common point. However, the intersection of the first two of these circles is  $q_{34}$  and thus  $q_{34}$  also lies on  $C(q_{24}, c_0^*, c_1^*)$ .

Interchanging the roles of 3 and 4 in this argument, we also deduce that  $q_{34}$  lies on  $C(q_{23}, c_0^*, c_1^*)$ . In other words,  $q_{34}, q_{24}, q_{23}$  and  $c_0^*, c_1^*$  all lie on the

same circle, as required.  $\square$

**Remark 4.2.** An identical theorem holds for a configuration of ten points  $q_{012}, \dots, q_{234}$  in  $Q^2$  labelled by the vertices of a  $g_3^5$  cell, namely that, if three of the octahedral faces are Clifford configurations, then so are the other two. This is not completely automatic, but follows from the antipodal symmetry of the Clifford condition (see Remark 2.2), as we need to take complementary indices to identify  $g_3^5$  and  $g_2^5$  cells.

**Remark 4.3.** An alternative formulation of Theorem 4.1 is that, if nine points, say  $q_{01}, \dots, q_{24}$ , have been chosen so that the two octahedra they form are Clifford configurations, then it is possible to choose the tenth point  $q_{34}$  in such a way that the three remaining octahedra, that contain it, are all Clifford configurations.

## 5. Four-dimensional consistency of Clifford lattices

We will now show that the  $g_2^5$  consistency expressed by Theorem 4.1 and Remark 4.3, together with the  $g_3^5$  consistency of Remark 4.2, provide the key to proving the integrability of Clifford lattices. Thus, Theorem 4.1 may be regarded as a geometric incarnation of the integrability of the dSKP equation (1.2). More precisely, we will use these two consistency conditions in the construction of a four-dimensional Clifford lattice  $q: \mathbb{Z}^4 \rightarrow Q^2$  (where the domain is really considered as an  $A_4$  lattice), with the property that restriction to any  $A_3$  sublattice in one of the five parallel families yields an ordinary three-dimensional Clifford lattice. Indeed, we will demonstrate that the two conditions guarantee that a solution propagates consistently from the same sort of two-dimensional Cauchy data that generates a three-dimensional Clifford lattice.

A Cauchy problem for any 6-point equation, such as dSKP, on (the octahedra of) an  $A_3$  lattice should lead to a unique solution on a given region from arbitrarily specified Cauchy data on the boundary of this region. Not all regions have this property, but, when it does hold, we will say that the region has a ‘Cauchy boundary’.

One example of a region with Cauchy boundary is (in  $\mathbb{Z}^3$  notation)

$$R(2_+, 3_+) = \{(m_1, m_2, m_3) \in \mathbb{Z}^3 : m_2 \geq 0, m_3 \geq 0\}.$$

Indeed, we claim that a solution in this region is uniquely determined by Cauchy data on the two half-hyperplanes

$$\begin{aligned} S(2_+, 3_0) &= \{(m_1, m_2, m_3) \in \mathbb{Z}^3 : m_2 \geq 0, m_3 = 0\} \\ S(2_0, 3_+) &= \{(m_1, m_2, m_3) \in \mathbb{Z}^3 : m_2 = 0, m_3 \geq 0\}. \end{aligned}$$

One way to see this is to project out the  $m_1$ -coordinate and interpret a lattice function  $q: \mathbb{Z}^3 \rightarrow Q^2$  as a function

$$\hat{q}: \mathbb{Z}^2 \rightarrow (Q^2)^{\mathbb{Z}}$$

on a two-dimensional lattice with coordinates  $(m_2, m_3)$ , whose ‘values’ are the  $\mathbb{Z}$  many values of the original three-dimensional lattice function with those two coordinates fixed. Thus, the 6-point equation, by which  $q_{23}$  is uniquely determined by  $q_1, q_2, q_3, q_{12}, q_{13}$ , becomes a 4-point equation by which  $\hat{q}_{23}$  is uniquely determined by  $\hat{q}, \hat{q}_2, \hat{q}_3$ . It is then a more familiar observation that a 4-point equation determines a solution on the quadrant  $\hat{R}(2_+, 3_+) \in \mathbb{Z}^2$  from Cauchy data on the two half-lines  $\hat{S}(2_+, 3_0)$  and  $\hat{S}(2_0, 3_+)$ .

**Remark 5.1.** A word of warning is needed here, as the formulation of the Cauchy problem in terms of  $\hat{q}$  gives a misleading impression of its symmetry. Regions like  $R(2_+, 3_+)$  and  $R(2_-, 3_-)$ , where two of the  $A_3$  coordinates have the same sign, do have Cauchy boundary, but, because of the asymmetric way in which the 6-point equation for  $q$  becomes a 4-point equation for  $\hat{q}$ , regions like  $R(2_+, 3_-)$  and  $R(2_-, 3_+)$ , where two coordinates have opposite signs, do not have Cauchy boundary. However, a region like  $R(2_+, 3_-)$  may be subdivided into two regions, such as  $R(1_+, 2_+, 3_-)$  and  $R(1_-, 2_+, 3_-)$ , which do have Cauchy boundary. Thus we can, in several ways, divide  $\mathbb{Z}^3$  into six regions to obtain a Cauchy problem for the whole lattice. For example, one could choose Cauchy data on the two planes  $S(2_0), S(3_0)$  and the two quarter-planes  $S(1_0, 2_+, 3_-), S(1_0, 2_-, 3_+)$ .

In this section, we will restrict attention to the simple partial Cauchy problem we first described; the one that can be analysed by reinterpreting  $q: \mathbb{Z}^3 \rightarrow Q^2$  as  $\hat{q}$  on  $\mathbb{Z}^2$ . Thus what we will prove is that a four-dimensional Clifford lattice  $q: \mathbb{Z}^4 \rightarrow Q^2$ , or  $\hat{q}$  on  $\mathbb{Z}^3$ , is uniquely determined on the region  $R(2_+, 3_+, 4_+)$  starting from Cauchy data on  $S(2_+, 3_0, 4_0)$ ,  $S(2_0, 3_+, 4_0)$  and  $S(2_0, 3_0, 4_+)$ . In terms of a 4-point equation on  $\hat{q}$ , this entails checking the familiar condition of ‘consistency around a cube’ as indicated in Figure 6: starting from  $\hat{q}, \hat{q}_2, \hat{q}_3, \hat{q}_4$  (black points) and using the equation to find

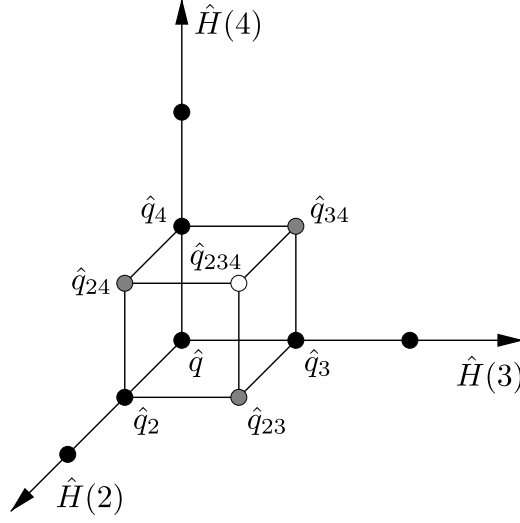


Figure 6: A projected Cauchy problem for four-dimensional Clifford lattices and consistency around the cube

$\hat{q}_{23}, \hat{q}_{24}, \hat{q}_{34}$  (grey points), one must check that  $\hat{q}_{234}$  (white point) is consistently determined by any one of the remaining equations.

With this in mind, we pursue the argument in  $\mathbb{Z}^4$  notation, using the 6-point Clifford condition and following Figure 4. Thus we use three Clifford conditions to determine the grey points

$$\begin{aligned} q_{23} & \text{ from } q_1, q_2, q_3, q_{12}, q_{13}, \\ q_{24} & \text{ from } q_1, q_2, q_4, q_{12}, q_{14}, \\ q_{34} & \text{ from } q_1, q_3, q_4, q_{13}, q_{14}, \end{aligned} \tag{5.1}$$

noting that the values on the right are all part of the Cauchy data (i.e. black points). We can then deduce from Theorem 4.1 (in  $\mathbb{Z}^4$  notation with the label 0 omitted) that the Clifford condition is also satisfied on the other two octahedra

$$q_2, q_3, q_4, q_{23}, q_{24}, q_{34} \quad \text{and} \quad q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}. \tag{5.2}$$

Now, a 1-shift of the above argument also gives us  $q_{123}, q_{124}, q_{134}$  (still grey points) and we may use the second of the octahedra in (5.2) and the 1-shift of the first to see that Clifford's condition is satisfied on

$$q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34} \quad \text{and} \quad q_{12}, q_{13}, q_{14}, q_{123}, q_{124}, q_{134}. \tag{5.3}$$

To finish, we need to know that we obtain the same answer by determining the white point  $q_{234}$  from  $q_{14}, q_{24}, q_{34}, q_{124}, q_{134}$ , or from  $q_{13}, q_{23}, q_{34}, q_{123}, q_{134}$ , or from  $q_{12}, q_{23}, q_{24}, q_{123}, q_{124}$ . This is a second application of Theorem 4.1, to the  $g_3^5$  cell of 10 points  $q_{012}, \dots, q_{234}$  (in  $A_4$  notation), relying on Remarks 4.2 and 4.3. Note the key role of the common octahedral face of the  $g_2^5$  and  $g_3^5$  cells that we use in the proof (cf. Figure 4).

Notice that every octahedron in each of the five families of parallel  $A_3$  sub-lattices in an  $A_4$  lattice is a face of some (in fact, unique)  $g_2^5$  cell and also of some  $g_3^5$  cell. Thus, since we have demonstrated that the Clifford condition holds on all five octahedral faces of the  $g_2^5$  (and  $g_3^5$ ) cell in every unit cube in  $R(2_+, 3_+, 4_+) \subset \mathbb{Z}^4$ , we have indeed shown that all five families of parallel  $A_3$  sub-lattices (within this region) form Clifford lattices.

## 6. Concluding Remarks

**Remark 6.1.** The Cauchy problem solved in Section 5 leads to a unique Clifford lattice on the region  $R(2_+, 3_+, 4_+) \subset \mathbb{Z}^4$ . As for  $\mathbb{Z}^3$  (cf. Remark 5.1), there are various ways of prescribing Cauchy data to obtain a Clifford lattice on the whole of  $\mathbb{Z}^4$ . For instance, a Clifford lattice  $q : \mathbb{Z}^4 \rightarrow Q^2$  is determined by Cauchy data on the three planes  $S(k_0, l_0)$  and six quarter-planes  $S(1_0, k_+, l_-, m_0)$ , for each choice of distinct  $k, l, m \in \{2, 3, 4\}$ .

Notice that, by virtue of Remark 5.1, the part of the above Cauchy data lying in the  $A_3$  sublattice  $R(4_0)$  is precisely what is needed to determine the three-dimensional Clifford lattice

$$q : R(4_0) \rightarrow Q^2. \quad (6.1)$$

Furthermore, the part lying in the  $A_3$  sublattice  $R(4_1)$ , namely the Cauchy data on the line  $L(2_0, 3_0, 4_1)$  and the two half-lines  $L(1_0, 2_-, 3_0, 4_1)$  and  $L(1_0, 2_0, 3_-, 4_1)$ , is precisely what is needed, in addition to the solution on  $R(4_0)$ , to determine the three-dimensional Clifford lattice

$$q : R(4_1) \rightarrow Q^2. \quad (6.2)$$

Thus, in the terminology of soliton theory [1], the Clifford lattice (6.2) can be interpreted as a Bäcklund transform of the Clifford lattice (6.1). In fact, this Bäcklund transformation is an analogue of the standard Bäcklund transformation of the Hirota-Miwa equation [8].

**Remark 6.2.** The existence of higher-dimensional Clifford lattices

$$q : A_n \rightarrow Q^2, \quad n \geq 5,$$

with the property that the restriction to any  $A_3$  sublattice yields an ordinary three-dimensional Clifford lattice, may be proved by considering the projected problem for the corresponding map

$$\hat{q} : \mathbb{Z}^{n-1} \rightarrow (Q^2)^{\mathbb{Z}},$$

as in Section 5. Indeed, it is well-known that, for purely combinatorial reasons, if  $\hat{q}$  is prescribed on the coordinate half-lines

$$\hat{S}(2_+, 3_0, \dots, n_0), \dots, \hat{S}(2_0, \dots, (n-1)_0, n_+),$$

then consistency around the cube, as indicated in Figure 6, guarantees that one obtains a unique map  $\hat{q} : R(2_+, \dots, n_+) \rightarrow (Q^2)^{\mathbb{Z}}$ , but to see that this actually yields (a sector of) an  $n$ -dimensional Clifford lattice, we must check that the Clifford condition holds on all  $\binom{n+1}{4}$  different types of  $g_2^4$  cells, i.e. octahedra.

Now,  $\binom{n-1}{2}$  of these octahedra are lifts of the quadrilaterals in  $\mathbb{Z}^{n-1}$  and so are Clifford configurations constructed as in (5.1). A further  $2\binom{n-1}{3}$  octahedra are of the type of (5.2), and so are Clifford configurations by an immediate application of the ‘three-implies-five property’ of Theorem 4.1. A typical one of the remaining  $\binom{n-1}{4}$  octahedra is given (in  $\mathbb{Z}^n$  notation) by

$$q_{23}, q_{24}, q_{25}, q_{34}, q_{35}, q_{45}$$

and this octahedron is a face of a  $g_2^5$  cell which has three other faces

$$q_2, q_3, q_4, q_{23}, q_{24}, q_{34}, \quad q_2, q_3, q_5, q_{23}, q_{25}, q_{35} \quad \text{and} \quad q_2, q_4, q_5, q_{24}, q_{25}, q_{45}$$

of the type of (5.2). Hence, a further application of the three-implies-five property guarantees that this remaining octahedron is also a Clifford configuration.

**Remark 6.3.** A degenerate case of a Clifford lattice  $q : \mathbb{Z}^3 \rightarrow Q^2$  occurs when the common point  $c^*$  is the same for every Clifford configuration  $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$  in the lattice. We may then take this point  $c^*$  to be the point at infinity in  $\mathbb{R}Q^2$  and interpret the lattice in the Euclidean plane, i.e.  $q : \mathbb{Z}^3 \rightarrow \mathbb{R}^2$ , with

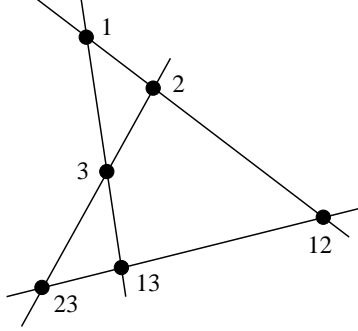


Figure 7: A Menelaus configuration

each of the triples  $(q_1, q_2, q_{12})$ ,  $(q_1, q_3, q_{13})$ ,  $(q_2, q_3, q_{23})$  and  $(q_{12}, q_{13}, q_{23})$  being collinear and thus forming a Menelaus configuration as in Figure 7. Such a degenerate Clifford lattice is known as a Menelaus lattice [10].

In this case, Theorem 4.1 degenerates to the statement that, if three of the octahedral faces are Menelaus configurations, then so are the other two. This is readily seen to be a formulation of the classical Desargues Theorem in the plane. Indeed, Figure 5 degenerates to the classical Desargues configuration with the five points  $c_1^*, \dots, c_5^*$  becoming the same point at infinity and all the circles becoming straight lines.

Although the argument used in proving Theorem 4.1 does not apply directly to this degenerate case, we can argue that sending  $c_2^*, c_3^*, c_4^*$  to infinity necessarily takes  $c_0^*, c_1^*$  to infinity as well and so the classical Desargues Theorem is obtained as a limiting case of Theorem 4.1.

In this degenerate case, the consistency argument of Section 5 works in largely the same way, except for a difference in the nature of the  $g_3^5$  version of Theorem 4.1. In a Menelaus lattice, while each  $g_2^5$  configuration degenerates to a classical Desargues configuration, each  $g_3^5$  configuration degenerates in a different way to a complete 5-line, as in Figure 8. In this case, the  $g_3^5$  version of Theorem 4.1 is effectively trivial, because of the way the 10 circles degenerate to 5 lines. More precisely, the point  $q_{234}$  simply needs to be chosen to be the intersection of the lines through  $q_{23}, q_{24}, q_{34}$  and  $q_{123}, q_{124}, q_{134}$  in order to create the three Menelaus configurations corresponding to the bottom three octahedra in Figure 4.

Thus we see that the classical Desargues Theorem is the essential geometric source of integrability for Menelaus lattices, even though it does not have the antipodal symmetry of its conformal generalization.

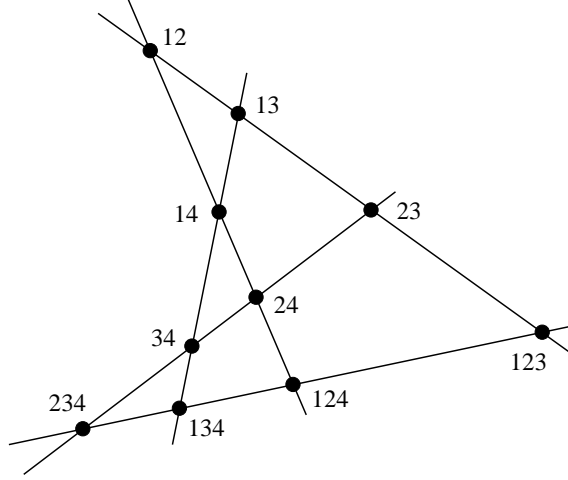


Figure 8: A complete 5-line

**Remark 6.4.** One point that has been highlighted in this paper is that, taking the  $A_n$  perspective on the integrability of 6-point equations like (1.2), requires an appropriate consistency condition to be satisfied on both  $g_2^5$  and  $g_3^5$  cells in the  $A_4$  lattice. In many cases (cf. Remark 4.2), these conditions are equivalent because of an antipodal symmetry inherent in the equations, although this is not automatically the case (cf. Remark 6.3).

For example, consider the discrete modified Kadomtsev-Petviashvili (dmKP) equation [11]

$$\frac{q_1 - q_2}{q_{12}} + \frac{q_2 - q_3}{q_{23}} + \frac{q_3 - q_1}{q_{13}} = 0, \quad (6.3)$$

for  $q: \mathbb{Z}^3 \rightarrow \mathbb{C}$ . The extension to  $\mathbb{Z}^4$  involves the additional three equations

$$\frac{q_1 - q_2}{q_{12}} + \frac{q_2 - q_4}{q_{24}} + \frac{q_4 - q_1}{q_{14}} = 0, \quad (6.4)$$

$$\frac{q_1 - q_3}{q_{13}} + \frac{q_3 - q_4}{q_{34}} + \frac{q_4 - q_1}{q_{14}} = 0, \quad (6.5)$$

$$\frac{q_2 - q_3}{q_{23}} + \frac{q_3 - q_4}{q_{34}} + \frac{q_4 - q_2}{q_{24}} = 0. \quad (6.6)$$

This system of four equations is known to be consistent in  $\mathbb{Z}^4$ . One way to see this is to observe that, not only is the fourth equation (6.6) an algebraic



consequence of the previous three, but so is a fifth equation

$$\frac{(q_{14} - q_{12})(q_{24} - q_{23})(q_{34} - q_{13})}{(q_{12} - q_{24})(q_{23} - q_{34})(q_{13} - q_{14})} = -1, \quad (6.7)$$

as noted in [2]. Thus, in  $A_n$  terminology, this new system of five equations is consistent around a  $g_2^5$  cell, as, in fact, any three imply the other two. One thing this example demonstrates is that it is not strictly necessary for the five equations on the five  $A_3$  sublattices of  $A_4$  to take the same form (cf. [2]).

Now, the consistency around a  $g_3^5$  cell requires the same three-implies-five property for (6.7) together with the 4-shift of (6.3), the 3-shift of (6.4), the 2-shift of (6.5) and the 1-shift of (6.6), which is *a priori* a different condition. However, the dmKP system also has a (slightly more subtle) antipodal symmetry, so that the  $g_3^5$  consistency is implied by the  $g_2^5$  consistency after simultaneously reflecting in the domain and inverting in the codomain:

$$(q, q_1, q_2, q_3, q_4, q_{12}, \dots) \mapsto (q^{-1}, q_1^{-1}, q_2^{-1}, q_3^{-1}, q_4^{-1}, q_{12}^{-1}, \dots),$$

where  $q_1^{-1}$  denotes a backward shift in the 1 direction, etc. Thus, if we also shift in the 1234 direction for clarity, (6.3) becomes

$$\frac{q_{234}^{-1} - q_{134}^{-1}}{q_{34}^{-1}} + \frac{q_{134}^{-1} - q_{124}^{-1}}{q_{14}^{-1}} + \frac{q_{124}^{-1} - q_{234}^{-1}}{q_{24}^{-1}} = 0$$

which rearranges to give the 4-shift of (6.3). The equations (6.4), (6.5), (6.6) behave similarly, while (6.7) is unchanged by the transformation, thereby yielding the required  $g_3^5$  consistency.

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